

## MATH 504 HOMEWORK 5

Due Monday, March 29.

**Problem 1.** Let  $M$  be a countable transitive model of ZFC and let  $\text{Add}(\omega, 1)$  be the poset of all partial finite functions from  $\omega$  to  $\{0, 1\}$ . Let  $G$  be a generic filter over  $M$ . As we did in class, define  $f^* : \omega \rightarrow \{0, 1\}$  to be  $f^* = \bigcup G$  and  $a = \{n \mid f^*(n) = 0\}$ . Recall that we proved in class that  $f^*$  is a total function with domain  $\omega$ .

- (1) Find two different  $\text{Add}(\omega, 1)$ -names in  $M$  for  $a$ , say  $\sigma$  and  $\tau$ , such that  $\sigma_G = \tau_G = a$ .
- (2) Show that  $a \notin M$ .

**Problem 2.** Show that  $G$  is a  $\mathbb{P}$ -generic filter over  $M$  (i.e. a filter that meets every dense  $D \subset \mathbb{P}$  in  $M$ ) iff  $G$  is a filter such that  $G$  meets every maximal antichain  $A \subset \mathbb{P}$  with  $A \in M$ .

**Problem 3.** Suppose that  $G$  is a  $\mathbb{P}$ -generic filter over  $M$  and  $p \in G$ .

- (1) Suppose that  $D \subset \mathbb{P}$  is **dense below**  $p$  i.e. for every  $q \leq p$ , there is  $r \leq q$  with  $r \in D$ . Show that  $G \cap D \neq \emptyset$ .
- (2) Let  $A \subset \mathbb{P}$  be an antichain such that for every  $q \in A, q \leq p$ , and for every  $r \leq p$ , there is  $q \in A$ , such that  $r, q$  are compatible i.e. they have a common extension. Show that  $G \cap A \neq \emptyset$ . Such a set  $A$  is called a maximal antichain below  $p$ .

**Problem 4.** Let  $M$  be a countable transitive model of ZFC and  $\mathbb{P} \in M$  be a poset. Suppose that  $\sigma$  and  $\tau$  are two  $\mathbb{P}$ -names in  $M$ , such that both  $\text{dom}(\sigma), \text{dom}(\tau) \subset \{\check{n} \mid n < \omega\}$ . Let

$$\pi = \{\langle \check{n}, p \rangle \mid (\exists q, r)(p \leq q \wedge p \leq r \wedge \langle \check{n}, q \rangle \in \sigma \wedge \langle \check{n}, r \rangle \in \tau)\}.$$

Show that  $\pi_G = \tau_G \cap \sigma_G$  for any generic filter  $G$  over  $M$

**Problem 5.** Let  $M$  be a countable transitive model of ZFC and  $\mathbb{P} \in M$  be a poset. Suppose that  $\sigma$  is a  $\mathbb{P}$ -names in  $M$ , such that both  $\text{dom}(\sigma) \subset \{\check{n} \mid n < \omega\}$ . Let

$$\pi = \{\langle \check{n}, p \rangle \mid (\forall q \in \mathbb{P})(\langle \check{n}, q \rangle \in \sigma \rightarrow q \perp p)\}.$$

Show that  $\pi_G = \omega \setminus \sigma_G$  for any generic filter  $G$  over  $M$  Hint: show that  $\{r \mid \exists p \geq r(\langle \check{n}, p \rangle \in \pi \vee \langle \check{n}, p \rangle \in \sigma)\}$  is dense.